

All unitary cubic curvature gravities in D dimensions

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We construct all the unitary cubic curvature gravity theories built on the contractions of the Riemann tensor in D -dimensional (anti)-de Sitter spacetimes. Our construction is based on finding the equivalent quadratic action for the general cubic curvature theory and imposing ghost and tachyon freedom, which greatly simplifies the highly complicated problem of finding the propagator of cubic curvature theories in constant curvature backgrounds. To carry out the procedure we have also classified all the unitary quadratic models. We use our general results to study the recently found cubic curvature theories using different techniques and the string generated cubic curvature gravity model. We also study the scattering in critical gravity and give its cubic curvature extensions.

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I. INTRODUCTION

Recently, a new purely cubic curvature theory in $(4 + 1)$ -dimensions with physically attractive properties was found in two separate works [1, 2]. In constructing this theory, different guiding principles were used. For example, in [1] the new cubic theory was uniquely singled out from the requirement that, just like in the higher curvature part of the $(2 + 1)$ -dimensional new massive gravity (NMG) [3], the trace of the field equations be second order and proportional to the Lagrangian itself, and that the field equations be second order for spherically symmetric solutions. On the other hand, the authors of [2] were searching for a cubic curvature theory with analytical, simple, spherically symmetric solutions and found the same theory, and coined it the “quasi-topological gravity”. The theory becomes more interesting once lower powers of curvature are added to it. D -dimensional extensions of the theory and its black hole solutions were found in [1, 2, 4, 5]. Also, extensive work on the model with regard to holography and the AdS/CFT appeared in [6–9]. Especially in [7, 9], the trace anomaly structure of the dual conformal field theory is studied and a (holographic) c -theorem *à la* [10] and [11] was proposed for both even and odd dimensions.

In this work, we will construct *all* the viable cubic curvature gravity theories with the guiding principle that the theory be unitarity around its (anti)-de Sitter [(A)dS] as well as its flat vacuum. We will find the most general cubic curvature theories that are perturbatively unitary in D dimensions. To check the unitarity of a given gravity theory around a fixed background, say, with metric \bar{g}_{ab} , one has to find its propagator, that is the $O(h^2)$ expansion of the theory where h_{ab} denotes the perturbation around the background. This is in general a highly complicated problem even for constant curvature backgrounds. But, the following observation somewhat simplifies the analysis: In generic D dimensions, if a higher curvature gravity theory is to be unitary, its propagator should, necessarily, reduce to the propagator of one of the following four unitary gravity models (the details and the unitarity regions of these theories will be given in Section II).

- I. The cosmological Einstein theory, $\frac{1}{\kappa}(R - 2\Lambda_0)$, with a massless spin-2 degree of freedom which is unitary for any Λ_0 and for $\kappa > 0$. (We take the mostly plus sign convention.)
- II. Einstein plus Gauss-Bonnet (EGB) theory, $\frac{1}{\kappa}(R - 2\Lambda_0) + \gamma(R_{abcd}^2 - 4R_{ab}^2 + R^2)$, with a massless spin-2 degree of freedom, but with an effective Newton’s constant $\frac{1}{\kappa_e} = \frac{1}{\kappa} + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma$, where Λ is the effective cosmological constant whose reality puts a constraint on the parameters. (Here and below, the effective Newton’s constant refers to the coefficient in front of the linearized Einstein tensor.) For unitarity $\kappa_e > 0$.
- III. Einstein plus scalar curvature theory, $\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2$, with a massless spin-2 degree of freedom and a massive spin-0 mode with an effective Newton’s constant $\frac{1}{\kappa_e} = \frac{1}{\kappa} + \frac{4\Lambda D}{D-2}\alpha$. For unitarity, $\kappa_e > 0$ and the mass of the scalar mode $m_s^2 = \frac{D-2}{4(D-1)\alpha\kappa_e} - \frac{2\Lambda D}{(D-1)(D-2)}$ should satisfy $m_s^2 > 0$ in dS, and the Breitenlohner-Freedman (BF) bound [12, 13]

$$m_s^2 \geq \frac{D-1}{2(D-2)}\Lambda,$$

in AdS.

- IV. Linear combination of the above three theories, $\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \gamma(R_{abcd}^2 - 4R_{ab}^2 + R^2)$, with a massless spin-2 degree of freedom and a massive spin-0 mode with an effective Newton’s constant $\frac{1}{\kappa_e} = \frac{1}{\kappa} + \frac{4\Lambda D}{D-2}\alpha + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma$. For unitarity, $\kappa_e > 0$ and the conditions on the mass of the scalar mode given in item III with the new κ_e .

Note that we do not consider the purely quadratic theories, since we would like to have the well-tested Einstein's gravity in the infrared region. But, our analysis can simply be extended to such models by considering $\kappa \rightarrow \infty$. Of course, the above models are not unitary for generic values of the parameters (especially in a curved background), there are constraints which we shall explore in more detail by looking at the one-particle exchange amplitude between two covariantly conserved sources [17]. We, also, do not consider the Fierz-Pauli massive gravity (with mass $m^2(h_{\mu\nu}^2 - h^2)$), even though it is unitary, it is not gauge invariant, so its propagator will not match any of the higher curvature models that we shall discuss. The classification of all the unitary cubic curvature theories that we shall present is exhaustive and devoid of any nonphysical or hard to justify assumptions. We have done a similar classification in three dimensions recently [18] where we have found all the bulk and boundary cubic curvature gravity theories.

The approach of analyzing the unitarity of the cubic curvature gravity theories by determining the quadratic curvature action which has the same propagator with the original cubic curvature action can also be used to construct the cubic curvature extensions of the critical gravity theory found in [14, 15]. As in the case of the unitary theories listed above, one can analyze the propagator structure of the critical theory by taking the proper limits of the one-particle exchange amplitude given in [17] and reveal the double pole structure of the critical gravity implying the logarithmic modes found in [24–26].

The layout of the paper is as follows: In Section II, we discuss the propagator and the spectra of quadratic curvature gravity theories in (A)dS and find the unitary regions that lead us to the above classification. In Section III, which is the bulk of this paper, we find the equivalent quadratic action that has the same free theory of a generic cubic curvature theory in D -dimensional (A)dS spacetime. In that section, we also discuss the unitarity of the previously found cubic curvature gravity theories mentioned above [1, 2, 4, 9] and the cubic curvature theory generated as an effective theory in the bosonic string theory [19]. In Appendix A, we give the one-particle exchange amplitude of the critical gravity and construct its cubic curvature extensions. Some details of the computations regarding the equivalent quadratic actions are delegated to the Appendix B.

II. PROPAGATOR STRUCTURE OF QUADRATIC CURVATURE GRAVITIES

Any higher curvature theory built on the powers of the Riemann tensor and its contractions in the form $f(R_{abcd})$ (but not its derivatives) has a propagator that has the *same structure* as the propagator of the following most general quadratic theory

$$I = \int d^D x \sqrt{-g} \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 + \gamma (R_{abcd}^2 - 4R_{ab}^2 + R^2) \right], \quad (1)$$

with possibly redefined parameters. Hence, to study the tree-level unitarity of a generic higher curvature theory, it suffices to understand the unitarity of (1): that is to find the constraints on the five parameters. In flat space with $\Lambda_0 = 0 = \Lambda$, it is well-known that for nonzero β , there is a massive ghost and the theory is not unitary [20]. (Note that Stelle's original result was only for $D = 4$, in fact, it is now known that a nonzero β is allowed in $D = 3$ [3]. For $D \geq 4$, Stelle's result is still valid in flat space.) For curved backgrounds, the unitarity analysis is actually quite involved and was carried out in [17], where the tree-level scattering amplitude between two background covariantly conserved sources was computed in the linearized version of the theory (1) augmented with a Fierz-Pauli (FP) mass term for the graviton. The FP mass helps to fix the gauge, and as long as the background is curved, after finding the propagator, it can be set to zero without introducing any van Dam-Veltman-Zakharov type discontinuity.

Since our subsequent discussions of the cubic curvature theories rest on the unitarity of the above quadratic theory, let us recapitulate the relevant analysis on the unitarity of (1). First of all, the two maximally symmetric vacua, with the Riemann tensor

$$\bar{R}_{abcd} = \frac{2\Lambda}{(D-1)(D-2)} (\bar{g}_{ac}\bar{g}_{bd} - \bar{g}_{ad}\bar{g}_{bc}), \quad (2)$$

of this theory satisfy

$$\frac{\Lambda - \Lambda_0}{2\kappa} + f\Lambda^2 = 0, \quad f \equiv (D\alpha + \beta) \frac{(D-4)}{(D-2)^2} + \gamma \frac{(D-3)(D-4)}{(D-1)(D-2)}. \quad (3)$$

One should keep in mind that γ enters into the picture only for $D \geq 5$. The linearized field equations for the metric fluctuation $h_{ab} \equiv g_{ab} - \bar{g}_{ab}$ around one of these constant curvature backgrounds become [21, 22]

$$T_{ab}(h) = \frac{1}{\kappa_e} \mathcal{G}_{ab}^L + (2\alpha + \beta) \left(\bar{g}_{ab} \bar{\square} - \bar{\nabla}_a \bar{\nabla}_b + \frac{2\Lambda}{D-2} \bar{g}_{ab} \right) R^L + \beta \left(\bar{\square} \mathcal{G}_{ab}^L - \frac{2\Lambda}{D-1} \bar{g}_{ab} R^L \right), \quad (4)$$

where we have defined the effective Newton's constant as

$$\frac{1}{\kappa_e} \equiv \frac{1}{\kappa} + \frac{4\Lambda D}{D-2} \alpha + \frac{4\Lambda}{D-1} \beta + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)} \gamma. \quad (5)$$

This expression was used in the list of the unitary theories given in the introduction. Here, $T_{ab}(h)$ contains the matter source τ_{ab} and all the higher order terms $O(h^{1+n})$. The linearized Einstein tensor, the Ricci tensor and the scalar curvature read

$$\mathcal{G}_{ab}^L = R_{ab}^L - \frac{1}{2} \bar{g}_{ab} R^L - \frac{2\Lambda}{D-2} h_{ab}, \quad (6)$$

$$R_{ab}^L = \frac{1}{2} \left(\bar{\nabla}^c \bar{\nabla}_a h_{bc} + \bar{\nabla}^c \bar{\nabla}_b h_{ac} - \bar{\square} h_{ab} - \bar{\nabla}_a \bar{\nabla}_b h \right), \quad R^L = -\bar{\square} h + \bar{\nabla}^a \bar{\nabla}^b h_{ab} - \frac{2\Lambda}{D-2} h. \quad (7)$$

It is important to realize that for all higher curvature gravity models, the building blocks in (4) will remain the same. Only the coefficients will be affected. For example, the n^{th} order higher curvature terms such as ηR^n , $\eta (R_{ab}^2)^{n/2}$, *etc.* will change $\frac{1}{\kappa_e}$ with an additional term proportional to $\eta \Lambda^{n-1}$, and will shift the coefficients $(2\alpha + \beta)$ and β with a term proportional to $\eta \Lambda^{n-2}$.

The tree-level scattering amplitude between two background covariantly conserved, $\bar{\nabla}_a T^{ab} = 0$, sources, as shown in Figure, is

$$A = \int d^D x \sqrt{-\bar{g}} T'_{ab}(x) h^{ab}(x). \quad (8)$$

Here, h_{ab} is the perturbation of the background metric created by one of the sources which is represented with T_{ab} . Our normalization of the amplitude is in such a way that in four dimensions for $\kappa = 8\pi G_N$ we have the usual Einstein equation. The one-graviton scattering amplitude between

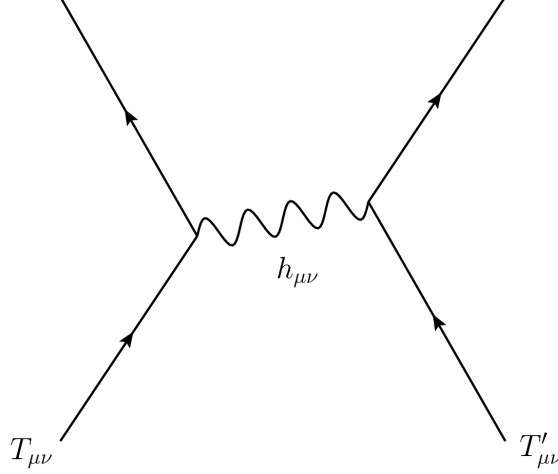


Figure 1: Tree-level scattering amplitude between two background covariantly conserved sources via the exchange of a graviton

the two covariantly conserved sources T_{ab} and T'_{ab} was calculated in [17] as

$$\begin{aligned}
\frac{A}{\kappa_e} = & 2T'_{ab} \left[\left(\kappa_e \beta \bar{\square} + 1 \right) \left(\Delta_L^{(2)} - \frac{4\Lambda}{D-2} \right) \right]^{-1} T^{ab} \\
& + \frac{2}{D-1} T' \left[\left(\kappa_e \beta \bar{\square} + 1 \right) \left(\bar{\square} + \frac{4\Lambda}{D-2} \right) \right]^{-1} T \\
& - \frac{4\Lambda}{(D-1)^2 (D-2)} T' \left[\left(\kappa_e \beta \bar{\square} + 1 \right) \left(\bar{\square} + \frac{4\Lambda}{D-2} \right) \right]^{-1} \left[\bar{\square} + \frac{2\Lambda D}{(D-1)(D-2)} \right]^{-1} T \\
& - \frac{2}{(D-1)(D-2)} T' \left[\kappa_e c \bar{\square} - 1 + \frac{2\Lambda D \kappa_e (c - \beta)}{(D-1)(D-2)} \right]^{-1} \left[\bar{\square} + \frac{2\Lambda D}{(D-1)(D-2)} \right]^{-1} T. \quad (9)
\end{aligned}$$

where $c \equiv \frac{4(D-1)\alpha + D\beta}{D-2}$ and $\Delta_L^{(2)}$ is the Lichnerowicz operator acting on a symmetric rank-2 tensor. To get (9), we have set the FP mass to zero as discussed above. To simplify the notation we have omitted the integral sign and the measure, and represented the Green's function as the inverse of the operator. Computation of (9) is a somewhat cumbersome problem, since one has to find the tensor Green's function corresponding to the equation $\mathcal{O}_{ab}^{cd} h_{cd} = T_{ab}$ where the operator \mathcal{O}_{ab}^{cd} has fourth order and second order derivatives, and a constant piece. The details are laid out in [17], therefore, we will not repeat it here, but just note that the best way seems to be to decompose h_{ab} into the transverse and traceless parts as

$$h_{ab} \equiv h_{ab}^{TT} + \bar{\nabla}_{(a} V_{b)} + \bar{\nabla}_a \bar{\nabla}_b \phi + \bar{g}_{ab} \psi, \quad (10)$$

then, gauge fix the equation with the help of a, say, FP mass term, and decompose T_{ab} in a similar way. This allows one to identify the physical parts of h_{ab} , and relate them to their sources. As we noted above, unless $D = 3$, β introduces a ghost. This becomes more transparent if one organizes

the $T - T'$ part of (9) and rewrites it as

$$\begin{aligned} \frac{A}{\kappa_e} = & 2T'_{ab} \left[\left(\kappa_e \beta \bar{\square} + 1 \right) \left(\Delta_L^{(2)} - \frac{4\Lambda}{D-2} \right) \right]^{-1} T^{ab} + \frac{2}{D-2} T' \left[\left(\kappa_e \beta \bar{\square} + 1 \right) \left(\bar{\square} + \frac{4\Lambda}{D-2} \right) \right]^{-1} T \\ & - \frac{2(\beta + c)}{c(D-1)(D-2)} T' \left[\left(\kappa_e \beta \bar{\square} + 1 \right) \left(\bar{\square} - m_s^2 \right) \right]^{-1} T \\ & + \frac{8\Lambda D \beta}{c(D-1)^2(D-2)^2} T' \left[\left(\kappa_e \beta \bar{\square} + 1 \right) \left(\bar{\square} - m_s^2 \right) \left(\bar{\square} + \frac{2\Lambda D}{(D-1)(D-2)} \right) \right]^{-1} T, \end{aligned} \quad (11)$$

where the mass of the scalar excitation is

$$m_s^2 = \frac{1}{c\kappa_e} - \frac{2\Lambda D}{(D-1)(D-2)} \left(1 - \frac{\beta}{c} \right). \quad (12)$$

If $\beta \neq 0$ for generic D , the poles can be separated and one can “see” the ghost. Since we have already studied the $D = 3$ theories in detail [18], we will consider $D > 3$. Therefore, we first set $\beta = 0$, then the apparent pole $\bar{\square} = -\frac{2\Lambda D}{(D-1)(D-2)}$ decouples. It is still a complicated task to explore the unitary regions of the remaining theory. One can somewhat simplify the analysis by relying on the unitarity of the pure cosmological Einstein-Hilbert theory whose unitarity has been checked with different techniques. One can write the one-particle exchange amplitude of the Einstein’s gravity, that is $\alpha = \gamma = 0$, as

$$A = 2\kappa \left[T'_{ab} \left(\Delta_L^{(2)} - \frac{4\Lambda}{D-2} \right)^{-1} T^{ab} + \frac{1}{D-2} T' \left(\bar{\square} + \frac{4\Lambda}{D-2} \right)^{-1} T \right], \quad (13)$$

which represents a unitary interaction via a massless spin-2 graviton for $\kappa > 0$. This expression is our canonical example which will guide us understand the spectrum and the unitarity of the other theories. Note that this result reduces to that of [23] in four dimensions.

Among the unitary theories listed in the introduction, the EGB theory augmented with the term αR^2 , which was classified as the fourth theory, contains the first three cases in the proper limits, therefore, let us start with the scattering amplitude of that theory which reads

$$\begin{aligned} A = & 2\kappa_e \left[T'_{ab} \left(\Delta_L^{(2)} - \frac{4\Lambda}{D-2} \right)^{-1} T^{ab} + \frac{1}{D-2} T' \left(\bar{\square} + \frac{4\Lambda}{D-2} \right)^{-1} T \right. \\ & \left. - \frac{1}{(D-1)(D-2)} T' \left(\bar{\square} - m_s^2 \right)^{-1} T \right], \end{aligned} \quad (14)$$

where the mass of the spin-0 mode given in (12) reduces to

$$m_s^2 = \frac{D-2}{4(D-1)\alpha\kappa_e} - \frac{2\Lambda D}{(D-1)(D-2)}, \quad (15)$$

and the effective Newton’s constant is

$$\frac{1}{\kappa_e} = \frac{1}{\kappa} + \frac{4\Lambda D}{D-2} \alpha + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)} \gamma. \quad (16)$$

Let us now study in some detail the unitarity regions and exploit the constraints on the parameters of this theory for all values and signs of Λ . For $\Lambda = 0$, the amplitude reduces to the simple flat space limit

$$A = 2\kappa \left[-T'_{ab} \left(\partial^2 \right)^{-1} T^{ab} + \frac{1}{D-2} T' \left(\partial^2 \right)^{-1} T - \frac{1}{(D-1)(D-2)} T' \left(\partial^2 - \frac{D-2}{4(D-1)\kappa\alpha} \right)^{-1} T \right], \quad (17)$$

which is unitary for $\kappa > 0$, and $m_s^2 = \frac{D-2}{4(D-1)\kappa\alpha} > 0$, and hence, $\alpha > 0$. As expected in flat space, the Gauss-Bonnet term does not play any role at the linearized level, therefore, γ is not restricted. On the other hand, in AdS ($\Lambda < 0$), the mass of the scalar excitation should satisfy the BF bound

$$m_s^2 \geq \frac{D-1}{2(D-2)}\Lambda, \quad (18)$$

which allows negative mass squared values. There is a wide region of parameters that yield a unitary theory. To require *attractive* gravity or nonghost behavior, we impose $\kappa_e > 0$, and we also demand that in the limit of vanishing Λ we have a unitary theory, and hence require $\kappa > 0$ and $\alpha > 0$. [Note that the latter two conditions can be removed to extend the unitarity regions.] All these conditions restrict γ to

$$\gamma < -\left(\frac{1}{4\kappa\Lambda} + \frac{\alpha D}{D-2}\right) \frac{(D-1)(D-2)}{(D-3)(D-4)}. \quad (19)$$

In the dS ($\Lambda > 0$) space, there is no BF bound on the scalar mode, instead one has $m_s^2 > 0$, and the unitarity region reads

$$\gamma > -\left(\frac{1}{4\kappa\Lambda} + \frac{\alpha D(D-4)}{(D-2)^2}\right) \frac{(D-1)(D-2)}{(D-3)(D-4)}. \quad (20)$$

Note that Λ is a function of Λ_0 , α and γ .

The scattering amplitude of the EGB theory, that is the second theory in our list, is obtained by setting $\alpha = 0$ in (14) and (15) which freezes the scalar mode and reduces the scattering amplitude to that of Einstein's gravity albeit with a modified gravitational constant. To get the $R - 2\Lambda_0 + \alpha R^2$ theory, one can simply set $\gamma = 0$ in (14) and (15).

One can also find the scattering amplitude for the critical gravity by using (9) and the discussion on this is given in Appendix A.

III. EQUIVALENT QUADRATIC ACTION

In the previous section, we have classified all the unitary quadratic gravity theories in D -dimensional (A)dS spacetime. Our main task, which is the purpose of this work, is now to extend this analysis to cubic curvature theories. In D dimensions, the most general cubic curvature action constructed from the contractions of the Riemann tensor, but not its derivatives, reads [28]

$$\begin{aligned} I = \int d^D x \sqrt{-g} & \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 + \gamma (R_{abcd}^2 - 4R_{ab}^2 + R^2) \right. \\ & + \left(a_1 R^{pqrs} R_p{}^t{}_r{}^u R_{qust} + a_2 R^{pqrs} R_{pq}{}^{tu} R_{rstu} + a_3 R^{pq} R^{rst}{}_p R_{rstq} \right. \\ & \left. \left. + a_4 R R^{pqrs} R_{pqrs} + a_5 R^{pq} R^{rs} R_{pqrs} + a_6 R^{pq} R_p{}^r{}_{qr} + a_7 R R^{pq} R_{pq} + a_8 R^3 \right) \right]. \end{aligned} \quad (21)$$

Altogether, we have 13 parameters and the main question is to find the ranges of these parameters for which the theory is perturbatively unitary around its (A)dS vacua (note that generically there will be 3 vacua). A direct way to answer this question is to compute the $O(h_{ab}^2)$ action and compare it with the unitary theories listed above. Computation of the $O(h_{ab}^2)$ action directly by computing

the relevant tensors to $O(h_{ab}^2)$ is a highly cumbersome analysis. To somewhat simplify the problem we will use a technique developed in [29] and extensively used in [30] to study the unitarity of the Born-Infeld (BI) type gravities, and in [18] to show the unitarity of the BI extension of NMG [31] and for the classification of all unitary cubic curvature gravities in three dimensions including the cubic curvature extension of NMG [32]. The technique consists of finding an *equivalent quadratic action* in curvature that has exactly the same propagator as (21). Moreover, as it will be clear below, this equivalent quadratic action also has the same vacua. This latter point is important since, with the help of the equivalent quadratic action, one avoids the derivation of the full nonlinear field equations to find the maximally symmetric vacua or determine the effective cosmological constant in terms of the 13 parameters of (21). The procedure of how one finds the equivalent quadratic action was described in [18] which we briefly repeat here. Consider a generic Lagrangian as a function of the Riemann tensor, $\mathcal{L} \equiv \sqrt{-g}F(R_{cd}^{ab})$. [We will choose our independent field to be the up-up-down-down Riemann tensor which will simplify the computations, since this choice does not introduce the background metric or its inverse. In fact, in the Appendix B, to further simplify the computations, we will consider the scalar curvature R and the Ricci tensor R_b^a to be the independent variables as well as the Riemann tensor R_{cd}^{ab} .] Finding the vacua and the $O(h^2)$ action means computing

$$\begin{aligned} \int d^D x \mathcal{L}(R_{cd}^{ab}) &= \int d^D x \mathcal{L}(\bar{R}_{cd}^{ab}) + \int d^D x \left[\frac{\delta \mathcal{L}}{\delta g^{ab}} \right]_{\bar{g}_{ab}} \delta g^{ab} \\ &+ \frac{1}{2} \int d^D x \delta g^{cd} \left[\frac{\delta \mathcal{L}}{\delta g^{cd} \delta g^{ab}} \right]_{\bar{g}_{ab}} \delta g^{ab} + \dots \end{aligned} \quad (22)$$

The first term in the right-hand side is irrelevant for our purposes, the second term gives the field equations and the third one gives the propagator. Finding an equivalent quadratic action means one finds a quadratic Lagrangian in curvature $\mathcal{L}_{\text{quad-equal}} \equiv \sqrt{-g}f_{\text{quad-equal}}(R_{cd}^{ab})$ that has the same $O(h^0)$, $O(h)$ and $O(h^2)$ expansions as the original Lagrangian \mathcal{L} . The equivalent quadratic Lagrangian can be obtained by expanding $F(R_{cd}^{ab})$ around the *yet to be found* constant curvature vacuum \bar{R}_{cd}^{ab} given in (2) up to second order as

$$f_{\text{quad-equal}}(R_{cd}^{ab}) \equiv \sum_{i=0}^2 \left[\frac{\partial^i F}{\partial (R_{cd}^{ab})^i} \right]_{\bar{R}_{cd}^{ab}} (R_{cd}^{ab} - \bar{R}_{cd}^{ab})^i. \quad (23)$$

Let us stress that $\sqrt{-g}f_{\text{quad-equal}}(R_{cd}^{ab})$ has the same $O(h^0)$, $O(h)$ and $O(h^2)$ expansions as the original Lagrangian. This simple, yet remarkable, result follows from the fact that all the, say, $O(h^2)$ terms of a given Lagrangian $\mathcal{L} \equiv \sqrt{-g}F(R_{cd}^{ab})$ are in the form

$$\begin{aligned} \mathcal{L}_{O(h^2)} &= \delta g^{cd} \left[\frac{\delta \mathcal{L}}{\delta g^{cd} \delta g^{ab}} \right]_{\bar{g}} \delta g^{ab} = [(\sqrt{-g})_{O(h^0)} + (\sqrt{-g})_{O(h^1)} + (\sqrt{-g})_{O(h^2)}] \\ &\times [F_{O(h^0)} + F_{O(h^1)} + F_{O(h^2)}] \\ &= (\sqrt{-g})_{O(h^0)} F_{O(h^2)} + (\sqrt{-g})_{O(h^1)} F_{O(h^1)} + (\sqrt{-g})_{O(h^2)} F_{O(h^0)}, \end{aligned} \quad (24)$$

where $(\sqrt{-g})_{O(h^n)}$ refers to the $O(h^n)$ expansion of $\sqrt{-g}$, F terms follow similarly. Here, all the terms involved in $[F_{O(h^0)} + F_{O(h^1)} + F_{O(h^2)}]$ are given as

$$F_{O(h^0)} + F_{O(h^1)} + F_{O(h^2)} = F(\bar{R}_{cd}^{ab}) + \left[\frac{\partial F}{\partial R_{cd}^{ab}} \right]_{\bar{R}_{cd}^{ab}} (R_{cd}^{ab})_{O(h)} + \left\{ \left[\frac{\partial F}{\partial R_{cd}^{ab}} \right]_{\bar{R}_{cd}^{ab}} (R_{cd}^{ab})_{O(h^2)} + \left[\frac{\partial^2 F}{\partial (R_{cd}^{ab})^2} \right]_{\bar{R}_{cd}^{ab}} \left[(R_{cd}^{ab})_{O(h)} \right]^2 \right\}. \quad (25)$$

As it is clear from the second line of this expression, the linear order in curvature also contributes to the $O(h^2)$ terms. Generically, all the terms up to order n in the expansion

$$\sum_{i=1}^n \left[\frac{\partial^i F}{\partial (R_{cd}^{ab})^i} \right]_{\bar{R}_{cd}^{ab}} (R_{cd}^{ab} - \bar{R}_{cd}^{ab})^i \quad (26)$$

contribute to $F_{O(h^n)}$.¹

Let us now apply these general ideas to (21). First of all, we do not need to consider the constant, the linear and the quadratic terms in the curvature, since they are already in the desired form. Let us rewrite the cubic action in such a way that R_{cd}^{ab} appears as the only independent variable

$$F(R_{cd}^{ab}) \equiv a_1 R_{rs}^{pq} R_{pt}^{ru} R_{qu}^{st} + a_2 R_{rs}^{pq} R_{pq}^{tu} R_{tu}^{rs} + a_3 R_q^p R_{tp}^{rs} R_{rs}^{tq} + a_4 R R_{rs}^{pq} R_{pq}^{rs} + a_5 R_q^p R_s^r R_{pr}^{qs} + a_6 R_q^p R_p^r R_r^q + a_7 R R_q^p R_p^q + a_8 R^3. \quad (27)$$

Here, note that we have chosen $R^{pqrs} R_p^t R_r^u R_{qust}$ specifically instead of $R^{pqrs} R_p^t R_r^u R_{qtsu}$ which was given in [28]. Because it is not possible to put the latter in the up-up-down-down form, $R^{pqrs} R_p^t R_r^u R_{qtsu} = R_{rs}^{pq} R_{pt}^{ru} R_q^{ts} R_u^{st}$. But, the two terms are related to each other as

$$R^{pqrs} R_p^t R_r^u R_{qtsu} = R^{pqrs} R_p^t R_r^u R_{qust} + \frac{1}{4} R^{pqrs} R_{pq}^{tu} R_{rstu}. \quad (28)$$

Leaving the details of the computation to the Appendix B, here we summarize the final results. The equivalent quadratic action that has the same free theory as (21) becomes

$$I = \int d^D x \sqrt{-g} \left[\frac{1}{\tilde{\kappa}} (R - 2\tilde{\Lambda}_0) + \tilde{\alpha} R^2 + \tilde{\beta} R_{ab}^2 + \tilde{\gamma} (R_{abcd}^2 - 4R_{ab}^2 + R^2) \right], \quad (29)$$

¹ In fact, another way of finding the equivalent quadratic action is to observe that at the desired order of $O(h^2)$, $(\text{Riem} - \bar{\text{Riem}})^3 = 0$. We thank a referee for pointing this fact.

with parameters

$$\frac{1}{\tilde{\kappa}} \equiv \frac{1}{\kappa} - \frac{12\Lambda^2}{(D-1)^2(D-2)^2} \times \left[(D-3)a_1 + 4a_2 + 2(D-1)(a_3 + Da_4) + (D-1)^2(a_5 + a_6 + Da_7 + D^2a_8) \right], \quad (30)$$

$$\tilde{\Lambda}_0 \equiv \frac{\tilde{\kappa}}{\kappa} \Lambda_0 + \frac{D\Lambda}{3(D-2)} \left(1 - \frac{\tilde{\kappa}}{\kappa} \right), \quad (31)$$

$$\tilde{\alpha} \equiv \alpha + \frac{2\Lambda}{(D-1)(D-2)} \times \left[3a_1 - 6a_2 - (D^2 - D - 4)a_4 + a_5 + (D-1)(-a_3 + 2a_7 + 3Da_8) \right], \quad (32)$$

$$\tilde{\beta} \equiv \beta + \frac{2\Lambda}{(D-1)(D-2)} \times \left[-9a_1 + 24a_2 + 4Da_3 + (2D-3)a_5 + (D-1)(4Da_4 + 3a_6 + Da_7) \right], \quad (33)$$

$$\tilde{\gamma} \equiv \gamma + \frac{2\Lambda}{(D-1)(D-2)} \left[-3a_1 + 6a_2 + (D-1)(a_3 + Da_4) \right]. \quad (34)$$

The equation that determines the effective cosmological constant is

$$\frac{\Lambda - \tilde{\Lambda}_0}{2\tilde{\kappa}} + \tilde{f}\Lambda^2 = 0, \quad \tilde{f} \equiv (D\tilde{\alpha} + \tilde{\beta}) \frac{(D-4)}{(D-2)^2} + \tilde{\gamma} \frac{(D-3)(D-4)}{(D-1)(D-2)}, \quad (35)$$

which, generically is a cubic order equation in Λ once $\tilde{\kappa}$ and \tilde{f} are explicitly written. To check the unitarity, one also needs the equivalent effective Newton's constant

$$\frac{1}{\tilde{\kappa}_e} \equiv \frac{1}{\tilde{\kappa}} + \frac{4\Lambda D}{D-2} \tilde{\alpha} + \frac{4\Lambda}{D-1} \tilde{\beta} + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)} \tilde{\gamma}. \quad (36)$$

The equations from (30) to (36) can be used to classify all the unitary cubic curvature theories. By choosing the parameters of the equivalent quadratic theory, one can reduce it to any one of those four unitary theories listed in the introduction and studied in Sec. II. For example, if $\tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} = 0$, the free theory of the cubic curvature gravity reduces to the cosmological Einstein theory with a massless spin-2 degree of freedom. These three conditions (which are linearly independent when α , β , and γ do not vanish) reduce the 13 parameters of the cubic theory to 10 parameters. The unitarity condition $\tilde{\kappa} > 0$ gives a constraint on these 10 parameters yielding a wide range of unitary cubic curvature theories. Let us note that as long as any one of the lower order parameters $\frac{1}{\kappa}$, Λ_0 , α , β , and γ do not vanish, the cubic equation (35) has at least *one* real root, meaning that the theory has a maximally symmetric vacuum with a nonzero curvature. Similarly, the cubic curvature theory can be reduced to the remaining four unitary theories, each time with only a couple of constraints on the parameters. This result classifies all the unitary cubic curvature theories in D dimensional (A)dS spacetimes. Therefore, it could also be used to check the unitarity of the existing cubic curvature gravity theories found with different guiding principles. Below we will study some examples such as the cubic curvature gravity obtained as an effective theory in the bosonic string theory [19], and the theories introduced in [1, 4] and [2, 7, 9].

Using (30-36), one can also find the cubic curvature extensions of the recently found critical gravity [14, 15] which is given in Appendix A.

A. The purely cubic theory

In [1, 2], the following cubic curvature theory was introduced

$$I = a \int d^5x \sqrt{-g} \left(-\frac{7}{6} R_{cd}^{ab} R_{bf}^{ce} R_{ae}^{df} - R_{ab}^{cd} R_{cd}^{be} R_e^a - \frac{1}{2} R^{pq} R^{rs} R_{pqrs} \right. \\ \left. + \frac{1}{3} R^{pq} R_p^r R_{qr} - \frac{1}{2} R R^{pq} R_{pq} + \frac{1}{12} R^3 \right). \quad (37)$$

The trace of the field equations coming from this action is proportional to the Lagrangian, and the action has just the first derivative of $g^{rr}(r)$ in the spherically symmetric ansatz. Note that there is no linear and quadratic terms in the curvature. One does not expect this homogeneous (in curvature) theory to have an (A)dS vacuum. This can easily be seen from the equivalent quadratic Lagrangian which follows from (29)

$$f_{\text{quad-equal}}(R_{cd}^{ab}) = -\frac{\Lambda^2}{12} \left(R - \frac{10\Lambda}{9} \right) + \frac{\Lambda}{12} (R_{abcd}^2 - 4R_{ab}^2 + R^2). \quad (38)$$

Using (35), one obtains $\Lambda = 0$ which says that the equivalent quadratic action (38) vanishes. Therefore, there is no propagator in this theory and there are no local degrees of freedom, unless one introduces linear and quadratic or zeroth order terms in the curvature. (It is likely that these theories have local degrees of freedom at the nonlinear level as in the case of Chern-Simons theories which lack local degrees of freedom at the quadratic level, but have nonzero degrees of freedom at the nonlinear level [33].) In fact, for any *purely* cubic curvature theory, unless the trace of the field equations vanishes identically, one does not have local degrees of freedom. [Taking the risk of being too pedantic, let us note that this fact follows from the vanishing of $O(h^2)$ expansion of *any* cubic or higher order curvature invariant around flat space.]

The generalization of (37) to D dimensions was given in [2, 4, 9] in different forms. Here, we choose the action given in [2]

$$\mathcal{Z}_D = a \int d^Dx \sqrt{-g} \left[R^{pqrs} R_p^t{}^u R_{qt su} + \frac{1}{(2D-3)(D-4)} \right. \\ \times \left(-3(D-2) R^{pq} R^{rst} R_{rstq} + \frac{3(3D-8)}{8} R R^{pqrs} R_{pqrs} \right. \\ \left. \left. + 3D R^{pq} R^{rs} R_{pqrs} + 6(D-2) R^{pq} R_p^r R_{qr} - \frac{3(3D-4)}{2} R R^{pq} R_{pq} + \frac{3D}{8} R^3 \right) \right]. \quad (39)$$

In fact, a second action was also given in [2]

$$\mathcal{Z}'_D = a \int d^Dx \sqrt{-g} \left[R^{pqrs} R_{pq}{}^{tu} R_{rstu} + \frac{1}{(2D-3)(D-4)} \left(-12(D^2-5D+5) R^{pq} R^{rst} R_{rstq} \right. \right. \\ \left. \left. + \frac{3}{2} (D^2-4D+2) R R^{pqrs} R_{pqrs} + 12(D-2)(D-3) R^{pq} R^{rs} R_{pqrs} \right. \right. \\ \left. \left. + 8(D-1)(D-3) R^{pq} R_p^r R_{qr} - 6(D-2)^2 R R^{pq} R_{pq} + \frac{1}{2} (D^2-4D+6) R^3 \right) \right], \quad (40)$$

which is related with the previous one as $\mathcal{Z}'_D = 2\mathcal{Z}_D + \frac{1}{4}\mathcal{X}_6$ where \mathcal{X}_6 is the six-dimensional Euler density

$$\mathcal{X}_6 = -8R^{pqrs} R_p^t{}^u R_{qt su} + 4R^{pqrs} R_{pq}{}^{tu} R_{rstu} - 24R^{pq} R^{rst} R_{rstq} \\ + 3R R^{pqrs} R_{pqrs} + 24R^{pq} R^{rs} R_{pqrs} + 16R^{pq} R_p^r R_{qr} - 12R R^{pq} R_{pq} + R^3. \quad (41)$$

To find the equivalent quadratic action for \mathcal{Z}_D , we first use (28) to see the Lagrangian as a function of R_{cd}^{ab} . Then, from (30, 31, 32, 33, 34), one arrives at the cosmological Einstein-Gauss-Bonnet theory

$$f_{\text{quad-equal}}(R_{cd}^{ab}) = -\frac{3(D-3)(3D^2-15D+16)\Lambda^2}{2(2D-3)(D-1)^2(D-2)}a\left(R - \frac{2D\Lambda}{3(D-2)}\right) \\ + \frac{3(3D^2-15D+16)\Lambda}{4(D-1)(D-2)(2D-3)}a\left(R_{abcd}^2 - 4R_{ab}^2 + R^2\right). \quad (42)$$

Again, one finds $\Lambda = 0$ from (35), and the theory has no propagating degrees of freedom. The equivalent quadratic action for \mathcal{Z}'_D is proportional to (42) with a proportionality constant $\frac{4(D^3-9D^2+26D-22)}{3D^2-15D+16}$. Therefore, as expected in the purely cubic curvature theories, it, too, has no local degrees of freedom. On the other hand, if we add lower powers of curvature to \mathcal{Z}_D or \mathcal{Z}'_D , breaking homogeneity, then we have theories with local degrees of freedom. As an example, let us consider \mathcal{Z}_D augmented with all the possible lower order terms

$$I = \int d^Dx \sqrt{-g} \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 + \gamma (R_{abcd}^2 - 4R_{ab}^2 + R^2) \right] + \mathcal{Z}_D. \quad (43)$$

The equivalent quadratic action has the following parameters

$$\frac{1}{\tilde{\kappa}} \equiv \frac{1}{\kappa} - \frac{3(D-3)(3D^2-15D+16)\Lambda^2}{2(2D-3)(D-1)^2(D-2)}a, \quad \tilde{\Lambda}_0 \equiv \frac{\tilde{\kappa}}{\kappa}\Lambda_0 + \frac{D\Lambda}{3(D-2)}\left(1 - \frac{\tilde{\kappa}}{\kappa}\right), \\ \tilde{\alpha} \equiv \alpha, \quad \tilde{\beta} \equiv \beta, \quad \tilde{\gamma} \equiv \gamma + \frac{3(3D^2-15D+16)\Lambda}{4(D-1)(D-2)(2D-3)}a, \quad (44)$$

where Λ is determined by (35), but the result is not particularly illuminating to depict. Let us give a simple specific example. Consider $D = 5$, $\alpha = \beta = \gamma = \Lambda_0 = 0$, then the three roots are given as

$$\Lambda = 0, \quad \Lambda_{\pm} = \pm 3\sqrt{\frac{7}{\kappa a}}. \quad (45)$$

To have an (A)dS vacuum, κa should be positive. For Λ_{\pm} , $\tilde{\kappa} = -\frac{\kappa}{8}$ and $\tilde{\kappa}_e = -\frac{\kappa}{2}$, therefore, to have $\kappa_e > 0$, we must choose $\kappa < 0$ and $a < 0$. The theory is unitary for these values, but it has the wrong sign bare Newton's constant. Of course, around $\Lambda = 0$, the theory is unitary for $\kappa > 0$, but the cubic terms do not contribute and the free theory is equivalent to the Einstein-Hilbert gravity. With nonvanishing Λ_0 , the picture changes, there are regions where the theory is unitary for $\kappa > 0$. Similar analysis can be done in D dimensions, but the punchline of this section is that the equivalent quadratic actions of \mathcal{Z}_D and \mathcal{Z}'_D are just the Einstein-Gauss-Bonnet theories.

B. Cubic theory of Oliva-Ray and Myers-Sinha

In [4], a cubic curvature theory was introduced using the requirement that the trace of the field equations is a second order equation of the metric whose cubic curvature action in our notation reads

$$I = \int d^Dx \sqrt{-g} \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \gamma (R_{abcd}^2 - 4R_{ab}^2 + R^2) + b_1 W_1 + b_2 W_2 + b_3 \chi_6 \right], \quad (46)$$

where $W_{1,2}$ are the inequivalent (for $D > 5$) contractions of three Weyl tensors

$$W_1 = C^{abcd}C_a{}^e{}_c{}^f C_{bedf}, \quad W_2 = C^{abcd}C_{ab}{}^{ef} C_{cdef}, \quad (47)$$

and χ_6 was given in (41). The same theory was introduced using AdS/CFT and the existence of a *simple* holographic c -function in [9]. To find the equivalent quadratic action of this theory and study the unitarity, there are two ways. The first way is to express W_1 and W_2 in terms of the tensor structures that appear in (27) and use (29). We have carried out this lengthy computation, but since the result is the same as a simpler computation that we shall lay out now, we will not present it here. The second, relatively simple, way is to realize that W_1 and W_2 do not contribute to the propagator in (A)dS or any conformally flat space as a matter of fact, and therefore, from (30, 31, 32, 33, 34) one can find the equivalent quadratic action as

$$\begin{aligned} f_{\text{quad-equal}}(R_{cd}^{ab}) = & -\frac{2\Lambda_0}{\kappa} + \frac{8D(D-3)(D-4)(D-5)\Lambda^3}{(D-1)^2(D-2)^2}b_3 \\ & + \left[\frac{1}{\kappa} - \frac{12(D-3)(D-4)(D-5)\Lambda^2}{(D-1)^2(D-2)}b_3 \right] R \\ & + \left[\gamma + \frac{6(D-4)(D-5)\Lambda}{(D-1)(D-2)}b_3 \right] (R_{abcd}^2 - 4R_{ab}^2 + R^2). \end{aligned} \quad (48)$$

Therefore, the free theory of (46) boils down to the EGB theory classified as the second unitary theory in the introduction. (The coefficient of the Gauss-Bonnet term can also vanish for a tuned value of b_3 yielding an interesting theory that we deal below.) The effective cosmological constant satisfies

$$\frac{\Lambda - \Lambda_0}{2\kappa} + \frac{\gamma\Lambda^2(D-3)(D-4)}{(D-1)(D-2)} + \frac{2b_3\Lambda^3(D-3)(D-4)(D-5)(D-6)}{(D-1)^2(D-2)^2} = 0, \quad (49)$$

which has at least one real root for any value of the parameters in generic $D > 6$ dimensions. As expected for $D \leq 6$, Euler density χ_6 does not contribute and for $D \leq 4$ the Gauss-Bonnet term does not contribute. The unitarity constraint

$$\frac{1}{\tilde{\kappa}_e} = \frac{1}{\kappa} + \frac{4\Lambda\gamma(D-3)(D-4)}{(D-1)(D-2)} + \frac{12b_3\Lambda^2(D-3)(D-4)(D-5)(D-6)}{(D-1)^2(D-2)^2} > 0, \quad (50)$$

which somewhat simplifies to

$$\frac{1}{\tilde{\kappa}_e} = \frac{3\Lambda_0}{\kappa\Lambda} - \frac{2}{\kappa} - \frac{2\Lambda\gamma(D-3)(D-4)}{(D-1)(D-2)} > 0, \quad (51)$$

restricts the parameters, but there is still a wide range of unitarity region. Let us specifically consider $D = 7$, and $\Lambda_0 = \gamma = 0$, for which one gets from (49)

$$\Lambda = 0, \quad \Lambda = \pm \frac{5}{2} \sqrt{-\frac{3}{2\kappa b_3}}. \quad (52)$$

The existence of a constant curvature vacuum requires $\kappa b_3 < 0$. Assuming $\kappa > 0$ to have the usual Einstein-Hilbert gravity in IR, one should choose $b_3 < 0$. However, this is in conflict with the UV unitarity, since $\tilde{\kappa}_e = -\frac{\kappa}{2} < 0$. Again, assuming nonzero Λ_0 or γ changes the picture and allows unitary regions which can be found from (49) and (51). Moreover, one can add αR^2 and βR_{ab}^2 to the action, and still have unitary regions.

Let us now consider the case for which (46) reduces to the pure Einstein-Hilbert gravity at the free-level. For this case, one should tune the coefficient of χ_6 (assuming $D > 6$)

$$b_3 = -\frac{\gamma(D-1)(D-2)}{6\Lambda(D-4)(D-5)}. \quad (53)$$

Then, (49) has two solutions

$$\Lambda_{\pm} = -\frac{3(D-1)(D-2)}{8\kappa\gamma(D-3)^2} \left(1 \pm \sqrt{1 + \frac{16\kappa\gamma\Lambda_0(D-3)^2}{3(D-1)(D-2)}} \right). \quad (54)$$

Further imposing the uniqueness of the vacuum $\Lambda_+ = \Lambda_-$, one has $\Lambda = 2\Lambda_0$ and a tuned Gauss-Bonnet coefficient

$$\gamma = -\frac{3(D-1)(D-2)}{16\kappa\Lambda_0(D-3)^2}. \quad (55)$$

The theory has an equivalent effective Newton's constant

$$\tilde{\kappa}_e = \tilde{\kappa} = \frac{4\kappa(D-3)}{(D-6)} > 0. \quad (56)$$

Summing up the result: the following cubic action has a unique (A)dS vacuum around which it has a unitary massless spin-2 mode

$$I = \int d^D x \sqrt{-g} \left[\frac{1}{\kappa} (R - 2\Lambda_0) - \frac{3(D-1)(D-2)}{16\kappa\Lambda_0(D-3)^2} (R_{abcd}^2 - 4R_{ab}^2 + R^2) \right. \\ \left. + b_1 W_1 + b_2 W_2 + \frac{(D-1)^2(D-2)^2}{64\kappa\Lambda_0^2(D-3)^2(D-4)(D-5)} \chi_6 \right]. \quad (57)$$

Furthermore, for $\Lambda_0 < 0$, the theory is also unitary in the boundary of AdS, since the free theory is exactly like the cosmological Einstein's gravity with modified parameters:

$$I_{\text{quad-equal}} = \int d^D x \sqrt{-g} \left[\frac{D-6}{4\kappa(D-3)} (R - 4\Lambda_0) \right]. \quad (58)$$

The fact that (57) and (58) have the same $O(h^0)$, $O(h)$ and $O(h^2)$ expansions is quite remarkable, and is a testament to the effectiveness of the tools laid out in this paper in studying the spectra and the vacua of higher curvature gravity theories.

C. Cubic gravity generated by string theory

In [19], using the three and four point scattering amplitudes of strings, it was shown that heterotic and type-II superstring theories do not produce any R^3 interactions, but the bosonic string has the following (in our basis) effective action

$$I = \frac{1}{\kappa} \int d^D x \sqrt{-g} \left[R + \frac{\alpha'}{4} (R_{abcd}^2 - 4R_{ab}^2 + R^2) + \frac{(\alpha')^2}{24} (-2R^{pqrs} R_p{}^t{}_r{}^u R_{qust} + R^{pqrs} R_{pq}{}^{tu} R_{rstu}) \right], \quad (59)$$

where α' is the usual inverse string tension. In this action, the coefficients of R_{ab}^2 and R^2 are not unambiguously determined by the computation of [19], but chosen for their simplicity. In our analysis, we will keep this choice; but, without much effort, one can of course consider these coefficients to be arbitrary. Note also that there is no bare cosmological constant. Let us now consider (59) to be our microscopic theory, forgetting its string theory origin as an effective theory, and study its spectrum and unitarity. Clearly, around flat space, it describes a massless spin-2

unitary excitation. Around its assumed (A)dS vacua, the equivalent quadratic action follows from (29)

$$I_{\text{quad-equal}} = \frac{1}{\kappa} \int d^D x \sqrt{-g} \left[-\frac{2\alpha'^2 \Lambda^3 D (D-5)}{3(D-1)^2 (D-2)^3} + \left(1 + \frac{\alpha'^2 \Lambda^2 (D-5)}{(D-1)^2 (D-2)^2} \right) R \right. \\ \left. + \frac{\alpha'}{4} (R_{abcd}^2 - 4R_{ab}^2 + R^2) + \frac{\alpha'^2 \Lambda}{2(D-1)(D-2)} (2R_{abcd}^2 - R_{ab}^2) \right]. \quad (60)$$

As before, Λ can be determined from the vacuum equation (35). Specifically, for the critical dimension of the bosonic string $D = 26$, we have $\Lambda = 0$, $\alpha' \Lambda \approx -2.37133$, and $\alpha' \Lambda \approx 26025.2$, but because of the last term in (60), the theory is not unitary in (A)dS. Even if one adds ²

$$\frac{\alpha'^2 \Lambda}{\kappa (D-1)(D-2)} \int d^D x \sqrt{-g} \left(R^2 - \frac{7}{2} R_{ab}^2 \right) \quad (61)$$

to the cubic action to turn the free part of the theory to the EGB theory with only a massless spin-2 excitation, the effective Newton's constant $\tilde{\kappa}_e$ becomes negative, meaning, one has a massless spin-2 ghost. Therefore, the theory is not unitary.

IV. CONCLUSION

We have systematically constructed all the perturbatively unitary cubic curvature gravity theories in D -dimensional (A)dS spacetimes. Our construction is based on finding an equivalent quadratic action that has the same vacua and the propagator as the original cubic theory. This equivalent quadratic action follows from a Taylor series expansion of the cubic action in powers of curvature up to second order. Using our general result, which was given in (29), we have studied the unitarity of the cubic curvature theories that appeared in the literature recently. These are the theories studied in [1, 2, 4, 6, 7, 9] that satisfy quite remarkable properties such as having second order equations and admitting simple c -functions. We have also studied the bosonic string generated cubic curvature gravity of [19] and found that it is nonunitary in (A)dS. We have also given unique vacuum cubic curvature extensions of the quadratic critical gravity theories. The tools we have presented here are very powerful in studying the free spectrum and the vacua of higher derivative gravity theories in (A)dS, and can be extended to any higher order. In fact, we have employed these tools in the infinite order Born-Infeld gravity theories [30], and the D -dimensional Lovelock gravities [34].

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² One might be worried about α'^2 appearing at the quadratic level, but Λ is $O(\frac{1}{\alpha'})$ for $D = 26$. In fact, one obtains $\Lambda = 0$, $\alpha' \Lambda \approx -2.41$, and $\alpha' \Lambda \approx -145.6$ (No dS vacuum!).

Appendix A: Scattering Amplitude and Cubic Curvature Extension of Critical Gravity

Critical gravity [14, 15] is defined by the action

$$I = \int d^D x \sqrt{-g} \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \gamma C_{abcd}^2 \right], \quad (\text{A1})$$

where the bare cosmological constant should be tuned to the value $\Lambda_0 = \frac{(D-1)(D-2)}{8\kappa\gamma(D-3)} = \Lambda$, and C_{abcd} is the Weyl tensor. The spectrum of the theory involves only massless spin-2 degree of freedom. The effective Newton's constant is given as $\kappa_e = \kappa \left(1 - \frac{D}{2}\right)$ which has the opposite sign compared to the bare one. As we will see, the propagator of this theory has a double pole at the zero on-shell mass.

1. Scattering in the critical gravity and its unitarity

Scattering in critical gravity theory is a somewhat subtle issue, this is because critical theory was defined for $T_{ab} = 0$, and the criticality strictly depends on $R_L = -\frac{2\Lambda}{D-2}h = 0$. If one wants to extend the critical theory to nonzero T_{ab} , then, apparently, one should restrict it to the case when the trace of the energy-momentum tensor is zero, $T = 0$. For example, in four dimensions, Maxwell theory satisfies this condition and provides an example of an extension of the critical theory with sources. First of all, in the parametrization of (1), the critical theory corresponds to [15]

$$\beta = -\frac{4\alpha(D-1)}{D}, \quad \Lambda = \Lambda_0 = -\frac{D}{8\kappa\alpha}, \quad \alpha = -\frac{\gamma D(D-3)}{(D-1)(D-2)}. \quad (\text{A2})$$

Then, for $T = 0$, the scattering amplitude reads

$$A = 2T'_{ab} \left[\left(\beta \bar{\square} + \frac{1}{\kappa_e} \right) \left(\Delta_L^{(2)} - \frac{4\Lambda}{D-2} \right) \right]^{-1} T^{ab}. \quad (\text{A3})$$

The Lichnerowicz operator $\Delta_L^{(2)}$ acting on a *traceless* rank-2 tensor φ_{ab} yields

$$\Delta_L^{(2)} \varphi_{ab} = \left(-\bar{\square} + \frac{4\Lambda D}{(D-1)(D-2)} \right) \varphi_{ab}, \quad (\text{A4})$$

and plugging the critical value of β in the form $\beta = -\frac{(D-1)(D-2)}{4\Lambda\kappa_e}$, one has the double pole structure

$$A = -\frac{2}{\beta} T'_{ab} \left[\left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right) \left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right) \right]^{-1} T^{ab}. \quad (\text{A5})$$

According to the old nomenclature, we have a degenerate Pais-Uhlenbeck oscillator; therefore, the two poles cannot be separated. With this observation, it is suggestive to think that the theory is free from the ghost, from which all the higher curvature theories (in $D > 3$) with an R_{ab}^2 term suffer, at the tree-level as long as $\beta > 0$ (for AdS, this requires $\kappa < 0$, and for dS $\kappa > 0$). However, in [16], it is shown that logarithmic modes mix, in the Hilbert space, with the homogeneous (Einstein) modes; and if one tries to restrict the physical space to homogeneous modes only, the physical space will only involve vacuum state.

Let us now consider the $T \neq 0$ case in the critical gravity. In this case, h cannot be zero, instead it is algebraically determined by the trace of the source

$$h = \frac{\kappa T}{\Lambda}, \quad (\text{A6})$$

which comes from (4). The degrees of freedom change and the scattering amplitude is no longer given as (A5), instead one has

$$\begin{aligned}
A = & \frac{2}{\beta} T'_{ab} \left[\left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right) \left(\Delta_L^{(2)} - \frac{4\Lambda}{D-2} \right) \right]^{-1} T^{ab} \\
& + \frac{2}{\beta(D-2)} T' \left[\left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right) \left(\bar{\square} + \frac{4\Lambda}{D-2} \right) \right]^{-1} T \\
& - \frac{1}{\Lambda\beta(D-2)} T' \left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right)^{-1} T \\
& + \frac{4D}{\beta(D-1)(D-2)^2} T' \left[\left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right) \left(\bar{\square} + \frac{2\Lambda D}{(D-1)(D-2)} \right) \right]^{-1} T.
\end{aligned} \tag{A7}$$

To see more explicitly the change in the degrees of freedom, let us compute the second order action in h_{ab} . The computation can be simplified by directly computing the linearized field equations

$$\mathcal{E}_{ab} \equiv \beta \left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right) \mathcal{G}_{ab}^L + \frac{\beta(D-2)}{2(D-1)} \left(\bar{g}_{ab} \bar{\square} - \bar{\nabla}_a \bar{\nabla}_b - \frac{2\Lambda}{D-2} \bar{g}_{ab} \right) R^L, \tag{A8}$$

and by integrating with the help of a reverse calculus of variations: $-\frac{1}{2} \int d^D x \sqrt{-\bar{g}} h^{ab} \mathcal{E}_{ab}$ where the overall factor comes from the self-adjointness of the involved operators. First, let us choose the gauge $\bar{\nabla}^a h_{ab} = \bar{\nabla}_b h$ as in [14], then $R^L = -\frac{2\Lambda}{D-2} h$. In this gauge, one has the linearized Ricci tensor

$$R_{ab}^L = \frac{1}{2} \left(\bar{\nabla}_a \bar{\nabla}_b h - \bar{\square} h_{ab} + \frac{4D\Lambda}{(D-1)(D-2)} h_{ab} - \frac{4\Lambda}{(D-1)(D-2)} \bar{g}_{ab} h \right), \tag{A9}$$

and the linearized Einstein tensor

$$\mathcal{G}_{ab}^L = \frac{1}{2} \left(\bar{\nabla}_a \bar{\nabla}_b h - \bar{\square} h_{ab} \right) + \frac{2\Lambda}{(D-1)(D-2)} h_{ab} + \frac{\Lambda(D-3)}{(D-1)(D-2)} \bar{g}_{ab} h. \tag{A10}$$

After using the gauge condition, one ends up with

$$\begin{aligned}
T_{ab} = & -\frac{\beta}{2} \left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right)^2 h_{ab} \\
& + \frac{\beta}{2} \bar{\square} \bar{\nabla}_a \bar{\nabla}_b h + \frac{\beta\Lambda(D-4)}{(D-1)(D-2)} \bar{\nabla}_a \bar{\nabla}_b h \\
& - \frac{\beta\Lambda}{(D-1)(D-2)} \left[\bar{\square} - \frac{2\Lambda(D^2-5D+8)}{(D-1)(D-2)} \right] \bar{g}_{ab} h.
\end{aligned} \tag{A11}$$

After integration by parts, the second order action in this source-coupled critical theory becomes

$$\begin{aligned}
I_{O(h^2)} = & -\frac{1}{2} \int d^D x \sqrt{-\bar{g}} \left\{ -\frac{\beta}{2} h^{ab} \left(\bar{\square} - \frac{4\Lambda}{(D-1)(D-2)} \right)^2 h_{ab} \right. \\
& \left. + \frac{\beta}{2} h \bar{\square}^2 h + \frac{\beta\Lambda(3D-7)}{(D-1)(D-2)} h \bar{\square} h + \frac{2\Lambda^2\beta(D^2-5D+8)}{(D-1)^2(D-2)^2} h^2 \right\}, \tag{A12}
\end{aligned}$$

which clearly shows that when $h \neq 0$, new degrees of freedom arise. To actually see these degrees of freedom and their masses, one has to do an *orthogonal* decomposition of the h_{ab} tensor as in (10), and further decompose h_{ab}^{TT} into “spatial” tensor, “spatial” vector, and “scalar” parts in a suitable form of the (A)dS metric. This computation is not needed for the purpose of this work. (For $D=3$, the computation was carried out for generic quadratic theory in [27].)

2. Cubic extensions of critical gravity

Using the equivalent quadratic action for cubic curvature theories given in (29-34), let us discuss the cubic curvature extensions of the critical theory found in [14, 15]. To get a critical cubic curvature gravity in (A)dS, the parameters of the equivalent quadratic action should satisfy the following conditions. The first condition is

$$\tilde{\beta} = -\frac{4\tilde{\alpha}(D-1)}{D}, \quad (\text{A13})$$

which removes the massive spin-0 mode, as long as

$$\frac{1}{\tilde{\kappa}} + 4\tilde{f}\Lambda \neq 0, \quad (\text{A14})$$

otherwise the trace of the linearized field equations is automatically zero, and therefore, R_L is left undetermined. As a second condition, the mass of the spin-2 excitation is set to zero:

$$\frac{1}{\tilde{\kappa}_e} + \frac{4\Lambda\tilde{\beta}}{(D-1)(D-2)} = 0, \quad (\text{A15})$$

which upon use of (A13) yields

$$0 = \frac{1}{\tilde{\kappa}} + 4\Lambda\tilde{f} + \frac{8\Lambda\tilde{\alpha}}{D}. \quad (\text{A16})$$

This second form makes the condition more transparent. In addition to these the vacuum equation has to be satisfied

$$\frac{\Lambda - \tilde{\Lambda}_0}{2\tilde{\kappa}} + \tilde{f}\Lambda^2 = 0, \quad \tilde{f} = [(D-1)(D-2)\tilde{\alpha} + D(D-3)\tilde{\gamma}] \frac{(D-4)}{D(D-1)(D-2)}. \quad (\text{A17})$$

In the case of the quadratic critical gravity, for which $\tilde{f} = f$ etc., (A16) defines the unique critical vacuum; therefore, one of the two roots of the second order vacuum equation is always noncritical. This could be remedied by imposing a unique critical vacuum which can be done in two ways: either one has coalescing roots or has $f = 0$. The first way contradicts (A16), but the second way leads to the unique vacuum critical theory

$$\beta = -\frac{4\alpha(D-1)}{D}, \quad \Lambda = \Lambda_0 = -\frac{D}{8\kappa\alpha}, \quad \alpha = -\frac{\gamma D(D-3)}{(D-1)(D-2)}. \quad (\text{A18})$$

The cubic curvature extension is more subtle even though its equivalent parameters satisfy the same equations (A13), (A14), (A16), and (A17). The subtlety arises from the fact that tilde variables depend on Λ . Therefore, (A16) is a second order equation in Λ and apparently there are two critical vacua. Moreover, (A17) is a cubic equation in Λ and those two critical vacua should be roots of this equation. But, the third root will always be noncritical (assuming the first two are real). Therefore, just like in the case of the quadratic critical theory, one should impose the unique vacuum condition. This again can be done in two different ways: either one has coalescing roots or has a linear equation. Again the first way is in contradiction with (A16). The second way leads to $\Lambda = \tilde{\Lambda}_0 = \Lambda_0$ and $\tilde{\kappa} = \kappa$. As in the quadratic critical gravity, Λ_0 is also tuned in terms of the other parameters. But, in this cubic critical gravity case, it satisfies a quadratic equation

$$\frac{1}{\kappa} + \frac{8\Lambda_0\tilde{\alpha}}{D} = 0, \quad (\text{A19})$$

where \tilde{a} is given in (32) (but, of course, two of a_i 's are eliminated in terms of the others). Therefore, for given parameters, say γ , a_1 , a_2 , a_3 , a_4 , a_5 , a_6 , there are *two* unique vacuum critical cubic curvature theories. For completeness, let us write the action

$$I = \int d^D x \sqrt{-g} \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \gamma C_{abcd}^2 + \left(a_1 R^{pqrs} R_p{}^t{}_r{}^u R_{qust} + a_2 R^{pqrs} R_{pq}{}^{tu} R_{rstu} + a_3 R^{pq} R^{rst}{}_p R_{rstq} \right. \right. \\ \left. \left. + a_4 R R^{pqrs} R_{pqrs} + a_5 R^{pq} R^{rs} R_{prqs} + a_6 R^{pq} R_p{}^r{}_{qr} + a_7 R R^{pq} R_{pq} + a_8 R^3 \right) \right], \quad (\text{A20})$$

with two of the parameters fixed as

$$a_7 = \frac{1}{D(D-1)(D-2)} \left[-3(D-6)a_1 - 24a_2 - 4(2D-3)a_3 \right. \\ \left. - 4D(D-1)a_4 - (2D-3)(D-2)a_5 - 3(D-1)(D-2)a_6 \right], \quad (\text{A21})$$

$$a_8 = \frac{1}{D^2(D-1)^2(D-2)} \left[2(D^2 - 8D + 6)a_1 + 4(5D-4)a_2 + 2(3D-4)(D-1)a_3 \right. \\ \left. + 2D^2(D-1)a_4 + (D-1)(D-2)^2a_5 + 2(D-1)^2(D-2)a_6 \right], \quad (\text{A22})$$

and Λ_0 is a root of the following equation

$$\frac{16\kappa D(D-3)[-3a_1 + 6a_2 + (D-1)(a_3 + Da_4)]}{(D-1)^2(D-2)^2} \Lambda_0^2 + \frac{8\kappa\gamma D(D-3)}{(D-1)(D-2)} \Lambda_0 - D = 0, \quad (\text{A23})$$

which gives a further restriction on the parameters coming from the reality of Λ_0 . It is interesting to see that one can have $\gamma = 0$, and generate a Einstein plus cubic curvature theory whose free theory is equivalent to the critical gravity, or one can have $\gamma \neq 0$ and extend the critical quadratic gravity to critical cubic gravity.

Appendix B: Calculation of Equivalent Quadratic Lagrangian Densities

To find the equivalent quadratic Lagrangian of a generic higher curvature gravity theory of the form $\mathcal{L} \equiv \sqrt{-g} F(R_{cd}^{ab})$, one needs the following expansion

$$f_{\text{quad-equal}}(R_{cd}^{ab}) = F(\bar{R}_{cd}^{ab}) + \left[\frac{\partial F}{\partial R_{ln}^{km}} \right]_{\bar{R}_{ln}^{km}} (R_{ln}^{km} - \bar{R}_{ln}^{km}) \\ + \frac{1}{2} \left[\frac{\partial^2 F}{\partial R_{xz}^{wy} \partial R_{ln}^{km}} \right]_{\bar{R}_{ln}^{km}} (R_{ln}^{km} - \bar{R}_{ln}^{km}) (R_{xz}^{wy} - \bar{R}_{xz}^{wy}). \quad (\text{B1})$$

Sometimes, computationally, it is easier to consider the Ricci tensor and the scalar as independent variables, and do the following expansion instead

$$\begin{aligned}
f_{\text{quad-equal}}(R, R_b^a, R_{cd}^{ab}) = & F(\bar{R}, \bar{R}_b^a, \bar{R}_{cd}^{ab}) + \left[\frac{\partial F}{\partial R} \right]_{\bar{R}} (R - \bar{R}) + \left[\frac{\partial F}{\partial R_j^i} \right]_{\bar{R}} (R_j^i - \bar{R}_j^i) \\
& + \left[\frac{\partial F}{\partial R_{ln}^{km}} \right]_{\bar{R}} (R_{ln}^{km} - \bar{R}_{ln}^{km}) \\
& + \frac{1}{2} \left[\frac{\partial^2 F}{\partial R^2} \right]_{\bar{R}} (R - \bar{R})^2 + \frac{1}{2} \left[\frac{\partial^2 F}{\partial R_j^i \partial R_l^k} \right]_{\bar{R}} (R_j^i - \bar{R}_j^i) (R_l^k - \bar{R}_l^k) \\
& + \frac{1}{2} \left[\frac{\partial^2 F}{\partial R_{xz}^{wy} \partial R_{ln}^{km}} \right]_{\bar{R}} (R_{ln}^{km} - \bar{R}_{ln}^{km}) (R_{xz}^{wy} - \bar{R}_{xz}^{wy}) \\
& + \left[\frac{\partial F}{\partial R \partial R_j^i} \right]_{\bar{R}} (R - \bar{R}) (R_j^i - \bar{R}_j^i) \\
& + \left[\frac{\partial F}{\partial R \partial R_{ln}^{km}} \right]_{\bar{R}} (R - \bar{R}) (R_{ln}^{km} - \bar{R}_{ln}^{km}) \\
& + \left[\frac{\partial F}{\partial R_j^i \partial R_{ln}^{km}} \right]_{\bar{R}} (R_j^i - \bar{R}_j^i) (R_{ln}^{km} - \bar{R}_{ln}^{km}).
\end{aligned} \tag{B2}$$

Of course (B1) and (B2) give the same results, but we will use the latter below. Let us consider all the 8 cubic curvature terms in (21).

1. Analysis of the three terms that do not involve the Riemann tensor

Let us first find the equivalent quadratic Lagrangian for the simplest term $F(R) \equiv R^3$ which reads

$$f_{\text{quad-equal}}(R) = \frac{8D^3\Lambda^3}{(D-2)^3} - \frac{12D^2\Lambda^2}{(D-2)^2}R + \frac{6D\Lambda}{D-2}R^2. \tag{B3}$$

The term constructed from the Ricci tensor alone $F(R_b^a) \equiv R_q^p R_p^r R_r^q$ has the following equivalent quadratic Lagrangian:

$$f_{\text{quad-equal}}(R_b^a) = \frac{8D\Lambda^3}{(D-2)^3} - \frac{12\Lambda^2}{(D-2)^2}R + \frac{6\Lambda}{D-2}R_{ab}^2. \tag{B4}$$

Finally, let us analyze the term involving the curvature scalar and the Ricci tensor, that is $F(R, R_b^a) \equiv R R_q^p R_p^q$:

$$f_{\text{quad-equal}}(R, R_b^a) = \frac{8D^2\Lambda^3}{(D-2)^3} - \frac{12D\Lambda^2}{(D-2)^2}R + \frac{4\Lambda}{D-2}R^2 + \frac{2D\Lambda}{D-2}R_{ab}^2. \tag{B5}$$

2. Separate analysis of the five terms involving Riemann tensor

a. The term $R_{rs}^{pq} R_{pt}^{ru} R_{qu}^{st}$:

Let $F(R_{cd}^{ab}) \equiv R_{rs}^{pq} R_{pt}^{ru} R_{qu}^{st}$. Then, the zeroth order part reads

$$F(\bar{R}_{cd}^{ab}) = \bar{R}_{rs}^{pq} \bar{R}_{pt}^{ru} \bar{R}_{qu}^{st} = \frac{8D(D-3)\Lambda^3}{(D-1)^2(D-2)^3}, \tag{B6}$$

and the first order part reads

$$\left[\frac{\partial F}{\partial R_{ln}^{km}} \right]_{\bar{R}} = 3 \bar{R}_{rk}^{pl} \bar{R}_{pm}^{rn} = \frac{12\Lambda^2}{(D-1)^2 (D-2)^2} \left[(D-2) \delta_k^l \delta_m^n + \delta_m^l \delta_k^n \right]. \quad (\text{B7})$$

In calculating the first order part, it is possible to take the derivatives with respect to the Riemann tensor such that the symmetries of the Riemann tensor are explicit. However, there is no need for such a care, because these symmetries will be taken into account upon multiplication with the difference terms $(R_{ln}^{km} - \bar{R}_{ln}^{km})$ and $(R_{xz}^{wy} - \bar{R}_{xz}^{wy})$ which satisfy the required symmetries, and the parts that do not obey the symmetries give zero contribution. The second order part reads

$$\begin{aligned} \left[\frac{\partial F}{\partial R_{xz}^{wy} \partial R_{ln}^{km}} \right]_{\bar{R}} &= 3 \left(\delta_y^l \delta_k^z \bar{R}_{wm}^{xn} + \bar{R}_{wk}^{xl} \delta_y^n \delta_m^z \right) \\ &= \frac{6\Lambda}{(D-1)(D-2)} \left(\delta_y^l \delta_k^z \delta_w^x \delta_m^n - \delta_y^l \delta_k^z \delta_m^x \delta_w^n + \delta_w^x \delta_k^l \delta_y^n \delta_m^z - \delta_k^x \delta_w^l \delta_y^n \delta_m^z \right). \end{aligned} \quad (\text{B8})$$

Therefore, the equivalent quadratic action for $F(R_{cd}^{ab}) \equiv R_{rs}^{pq} R_{pt}^{ru} R_{qu}^{st}$ becomes

$$\begin{aligned} f_{\text{quad-equal}}(R_{cd}^{ab}) &= \frac{8D(D-3)\Lambda^3}{(D-1)^2 (D-2)^3} - \frac{12(D-3)\Lambda^2}{(D-1)^2 (D-2)^2} R \\ &\quad - \frac{6\Lambda}{(D-1)(D-2)} R_{abcd}^2 + \frac{6\Lambda}{(D-1)(D-2)} R_{ab}^2. \end{aligned} \quad (\text{B9})$$

b. The term $R_{rs}^{pq} R_{pq}^{tu} R_{tu}^{rs}$:

Let $F(R_{cd}^{ab}) \equiv R_{rs}^{pq} R_{pq}^{tu} R_{tu}^{rs}$. Then, the zeroth order, the first order and the second order parts read

$$F(\bar{R}_{cd}^{ab}) = \bar{R}_{rs}^{pq} \bar{R}_{pq}^{tu} \bar{R}_{tu}^{rs} = \frac{32D\Lambda^3}{(D-1)^2 (D-2)^3}, \quad (\text{B10})$$

$$\left[\frac{\partial F}{\partial R_{ln}^{km}} \right]_{\bar{R}} = 3 \bar{R}_{km}^{tu} \bar{R}_{tu}^{ln} = \frac{24\Lambda^2}{(D-1)^2 (D-2)^2} \left(\delta_k^l \delta_m^n - \delta_k^n \delta_m^l \right), \quad (\text{B11})$$

$$\begin{aligned} \left[\frac{\partial F}{\partial R_{xz}^{wy} \partial R_{ln}^{km}} \right]_{\bar{R}} &= 3 \left(\delta_k^x \delta_m^z \bar{R}_{wy}^{ln} + \delta_w^l \delta_y^n \bar{R}_{km}^{xz} \right) \\ &= \frac{6\Lambda}{(D-1)(D-2)} \left(\delta_k^x \delta_m^z \delta_w^l \delta_y^n - \delta_k^x \delta_m^z \delta_y^l \delta_w^n + \delta_w^l \delta_y^n \delta_k^x \delta_m^z - \delta_w^l \delta_y^n \delta_m^x \delta_k^z \right), \end{aligned} \quad (\text{B12})$$

yielding an equivalent quadratic action

$$f_{\text{quad-equal}}(R_{cd}^{ab}) = \frac{32D\Lambda^3}{(D-1)^2 (D-2)^3} - \frac{48\Lambda^2}{(D-1)^2 (D-2)^2} R + \frac{12\Lambda}{(D-1)(D-2)} R_{abcd}^2. \quad (\text{B13})$$

c. The term $R_q^p R_{tp}^{rs} R_{rs}^{tq}$:

Let $F(R_b^a, R_{cd}^{ab}) \equiv R_q^p R_{tp}^{rs} R_{rs}^{tq}$. Then, the relevant derivatives are found as

$$F(\bar{R}_b^a, \bar{R}_{cd}^{ab}) = \bar{R}_q^p \bar{R}_{tp}^{rs} \bar{R}_{rs}^{tq} = \frac{16D\Lambda^3}{(D-1)(D-2)^3}, \quad (\text{B14})$$

$$\left[\frac{\partial F}{\partial R_j^i} \right]_{\bar{R}} = \bar{R}_{ti}^{rs} \bar{R}_{rs}^{tj} = \frac{8\Lambda^2}{(D-1)(D-2)^2} \delta_i^j, \quad (\text{B15})$$

$$\left[\frac{\partial F}{\partial R_{ln}^{km}} \right]_{\bar{R}} = \bar{R}_q^n \bar{R}_{km}^{lq} + \bar{R}_m^p \bar{R}_{kp}^{ln} = \frac{8\Lambda^2}{(D-1)(D-2)^2} (\delta_k^l \delta_m^n - \delta_m^l \delta_k^n), \quad (\text{B16})$$

$$\left[\frac{\partial^2 F}{\partial R_{xz}^{wy} \partial R_{ln}^{km}} \right]_{\bar{R}} = \bar{R}_y^n \delta_w^l \delta_k^x \delta_m^z + \bar{R}_m^z \delta_k^x \delta_w^l \delta_y^n = \frac{4\Lambda}{D-2} \delta_w^l \delta_y^n \delta_k^x \delta_m^z, \quad (\text{B17})$$

$$\begin{aligned} \left[\frac{\partial F}{\partial R_j^i \partial R_{ln}^{km}} \right]_{\bar{R}} &= \delta_i^n \bar{R}_{km}^{lj} + \delta_m^j \bar{R}_{ki}^{ln} \\ &= \frac{2\Lambda}{(D-1)(D-2)} \delta_i^n (\delta_k^l \delta_m^j - \delta_m^l \delta_k^j) + \frac{2\Lambda}{(D-1)(D-2)} \delta_m^j (\delta_k^l \delta_i^n - \delta_i^l \delta_k^n), \end{aligned} \quad (\text{B18})$$

summing up to

$$\begin{aligned} f_{\text{quad-equal}}(R_b^a, R_{cd}^{ab}) &= \frac{16D\Lambda^3}{(D-1)(D-2)^3} - \frac{24\Lambda^2}{(D-1)(D-2)^2} R \\ &\quad + \frac{2\Lambda}{D-2} R_{abcd}^2 + \frac{8\Lambda}{(D-1)(D-2)} R_{ab}^2. \end{aligned} \quad (\text{B19})$$

d. The term $RR_{rs}^{pq} R_{pq}^{rs}$:

Let $F(R, R_{cd}^{ab}) \equiv RR_{rs}^{pq} R_{pq}^{rs}$. Then, computing the relevant derivatives

$$F(\bar{R}, \bar{R}_{cd}^{ab}) = \bar{R} \bar{R}^{pqrs} \bar{R}_{pqrs} = \frac{16D^2\Lambda^3}{(D-1)(D-2)^3}, \quad (\text{B20})$$

$$\left[\frac{\partial F}{\partial R} \right]_{\bar{R}} = \bar{R}_{rs}^{pq} \bar{R}_{pq}^{rs} = \frac{8D\Lambda^2}{(D-1)(D-2)^2}, \quad (\text{B21})$$

$$\left[\frac{\partial F}{\partial R_{ln}^{km}} \right]_{\bar{R}} = 2\bar{R} \bar{R}_{km}^{ln} = \frac{8D\Lambda^2}{(D-1)(D-2)^2} (\delta_k^l \delta_m^n - \delta_m^l \delta_k^n) \quad (\text{B22})$$

$$\left[\frac{\partial^2 F}{\partial R_{xz}^{wy} \partial R_{ln}^{km}} \right]_{\bar{R}} = 2\bar{R} \delta_w^l \delta_y^n \delta_k^x \delta_m^z = \frac{4D\Lambda}{D-2} \delta_w^l \delta_y^n \delta_k^x \delta_m^z, \quad (\text{B23})$$

$$\left[\frac{\partial F}{\partial R \partial R_{ln}^{km}} \right]_{\bar{R}} = 2\bar{R}_{km}^{ln} = \frac{4\Lambda}{(D-1)(D-2)} \left(\delta_k^l \delta_m^n - \delta_m^l \delta_k^n \right), \quad (\text{B24})$$

the equivalent quadratic Lagrangian becomes

$$\begin{aligned} f_{\text{quad-equal}}(R, R_{cd}^{ab}) &= \frac{16D^2\Lambda^3}{(D-1)(D-2)^3} - \frac{24D\Lambda^2}{(D-1)(D-2)^2} R \\ &\quad + \frac{2D\Lambda}{D-2} R_{abcd}^2 + \frac{8\Lambda}{(D-1)(D-2)} R^2. \end{aligned} \quad (\text{B25})$$

e. The term $R_q^p R_s^r R_{pr}^{qs}$.

Let $F(R_b^a, R_{cd}^{ab}) \equiv R_q^p R_s^r R_{pr}^{qs}$. Then, computing the relevant derivatives

$$F(\bar{R}_b^a, \bar{R}_{cd}^{ab}) = \bar{R}_q^p \bar{R}_s^r \bar{R}_{pr}^{qs} = \frac{8D\Lambda^3}{(D-2)^3}, \quad (\text{B26})$$

$$\left[\frac{\partial F}{\partial R_j^i} \right]_{\bar{R}} = 2\bar{R}_s^r \bar{R}_{ir}^{js} = \frac{8\Lambda^2}{(D-2)^2} \delta_i^j, \quad (\text{B27})$$

$$\left[\frac{\partial F}{\partial R_{ln}^{km}} \right]_{\bar{R}} = \bar{R}_k^l \bar{R}_m^n = \frac{4\Lambda^2}{(D-2)^2} \delta_k^l \delta_m^n, \quad (\text{B28})$$

$$\left[\frac{\partial^2 F}{\partial R_j^i \partial R_l^k} \right]_{\bar{R}} = 2\bar{R}_{ik}^{jl} = \frac{4\Lambda}{(D-1)(D-2)} \left(\delta_i^j \delta_k^l - \delta_k^j \delta_i^l \right), \quad (\text{B29})$$

$$\left[\frac{\partial F}{\partial R_j^i \partial R_{ln}^{km}} \right]_{\bar{R}} = 2\bar{R}_m^n \delta_k^j \delta_i^l = \frac{4\Lambda}{D-2} \delta_m^n \delta_k^j \delta_i^l, \quad (\text{B30})$$

the equivalent quadratic Lagrangian becomes

$$f_{\text{quad-equal}}(R_b^a, R_{cd}^{ab}) = \frac{8D\Lambda^3}{(D-2)^3} - \frac{12\Lambda^2}{(D-2)^2} R + \frac{2\Lambda}{(D-1)(D-2)} R^2 + \frac{2(2D-3)\Lambda}{(D-1)(D-2)} R_{ab}^2. \quad (\text{B31})$$

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- [1] J. Oliva and S. Ray, Class. Quant. Grav. **27**, 225002 (2010).
 - [2] R. C. Myers and B. Robinson, JHEP **1008**, 067 (2010).
 - [3] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009).
 - [4] J. Oliva and S. Ray, Phys. Rev. D **82**, 124030 (2010).
 - [5] W. G. Brenna, M. H. Dehghani, R. B. Mann, Phys. Rev. D **84**, 024012 (2011).
 - [6] R. C. Myers, M. F. Paulos and A. Sinha, JHEP **1008**, 035 (2010).
 - [7] R. C. Myers and A. Sinha, Phys. Rev. D **82**, 046006 (2010).
 - [8] A. Sinha, Class. Quant. Grav. **28**, 085002 (2011)

- [9] R. C. Myers and A. Sinha, JHEP **1101**, 125 (2011).
- [10] A. B. Zamolodchikov, JETP Lett. **43**, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. **43**, 565 (1986)].
- [11] J. L. Cardy, Phys. Lett. B **215**, 749 (1988).
- [12] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B **115**, 197 (1982).
- [13] L. Mezincescu and P. K. Townsend, Annals Phys. **160**, 406 (1985).
- [14] H. Lu and C. N. Pope, Phys. Rev. Lett. **106**, 181302 (2011).
- [15] S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman and B. Tekin, Phys. Rev. D **83**, 061502 (2011).
- [16] M. Porrati and M. M. Roberts, “Ghosts of Critical Gravity,” arXiv:1104.0674 [hep-th].
- [17] I. Gullu and B. Tekin, Phys. Rev. D **80**, 064033 (2009).
- [18] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D **83**, 024033 (2011).
- [19] R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B **185**, 52 (1987).
- [20] K. S. Stelle, Phys. Rev. D **16**, 953 (1977); Gen. Rel. Grav. **9**, 353 (1978).
- [21] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003).
- [22] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002).
- [23] M. Porrati, Phys. Lett. B **498**, 92 (2001).
- [24] M. Alishahiha and R. Fareghbal, Phys. Rev. D **83**, 084052 (2011).
- [25] I. Gullu, M. Gurses, T. C. Sisman and B. Tekin, Phys. Rev. D **83**, 084015 (2011).
- [26] E. A. Bergshoeff, O. Hohm, J. Rosseel and P. K. Townsend, Phys. Rev. D **83**, 104038 (2011).
- [27] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D **81**, 104017 (2010).
- [28] S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins, Class. Quant. Grav. **9**, 1151 (1992).
- [29] A. Hindawi, B. A. Ovrut and D. Waldram, Phys. Rev. D **53**, 5597 (1996).
- [30] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D **82**, 124023 (2010).
- [31] I. Gullu, T. C. Sisman and B. Tekin, Class. Quant. Grav. **27**, 162001 (2010).
- [32] A. Sinha, JHEP **1006**, 061 (2010).
- [33] M. Banados, L. J. Garay and M. Henneaux, Nucl. Phys. B **476**, 611 (1996).
- [34] T. C. Sisman, I. Gullu and B. Tekin, to appear.